Variable Annuities with Guaranteed Minimum Withdrawal Benefits

Patrick Cheridito and Peiqi Wang

ETH Zurich and Princeton University

SAA Annual Meeting 2016 in Fribourg

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
- Variable annuities with protection features have gained popularity

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
- Variable annuities with protection features have gained popularity

VA+GMxB: Variable annuities with benefits

- GMDB: Guaranteed minimum death benefits

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
- Variable annuities with protection features have gained popularity

VA+GMxB: Variable annuities with benefits

- GMDB: Guaranteed minimum death benefits
- GMAB: Guaranteed minimum accumulation benefits

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
- Variable annuities with protection features have gained popularity

VA+GMxB: Variable annuities with benefits

- GMDB: Guaranteed minimum death benefits
- GMAB: Guaranteed minimum accumulation benefits
- GMIB: Guaranteed minimum income benefits

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

- Exposure to income and longevity risk shifts from employer to the employed
- Variable annuities with protection features have gained popularity

VA+GMxB: Variable annuities with benefits

- GMDB: Guaranteed minimum death benefits
- GMAB: Guaranteed minimum accumulation benefits
- GMIB: Guaranteed minimum income benefits
- GMWB: Guaranteed minimum withdrawal benefits

Variable annuities with GMxB

Change from defined-benefit to defined-contribution pension plans

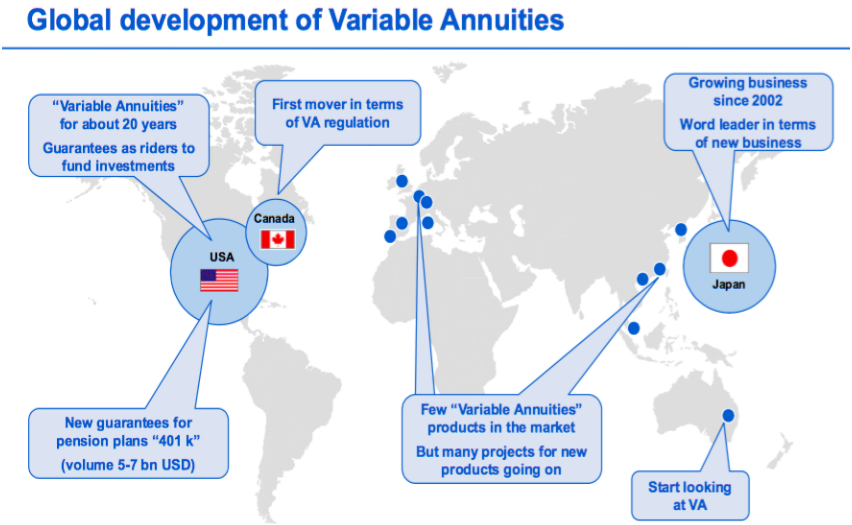
- Exposure to income and longevity risk shifts from employer to the employed
- Variable annuities with protection features have gained popularity

VA+GMxB: Variable annuities with benefits

- GMDB: Guaranteed minimum death benefits
- GMAB: Guaranteed minimum accumulation benefits
- GMIB: Guaranteed minimum income benefits
- GMWB: Guaranteed minimum withdrawal benefits

(See Gerold Studer's talk at the SAA Annual Meeting 2010 for an overview)

From Gerold Studer's talk at the SAA Annual Meeting 2010



Increasing popularity of GMWB with high water mark features (ratchets)

Description on the Vanguard Group website of one of their VA+GMWB

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically
- Gain insights into ...
  - how the price depends on the risk free rate, volatility, ...

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically
- Gain insights into ...
  - how the price depends on the risk free rate, volatility, ...
  - the worst case policy holder behavior from the point of view of the issuer

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically
- Gain insights into ...
  - how the price depends on the risk free rate, volatility, ...
  - the worst case policy holder behavior from the point of view of the issuer
  - how insurance companies should set the fee structure

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically
- Gain insights into ...
  - how the price depends on the risk free rate, volatility, ...
  - the worst case policy holder behavior from the point of view of the issuer
  - how insurance companies should set the fee structure
  - how surrender penalties can be set to disincentivize early surrender

What's the plan for this talk?

- Formulate a tractable model for a VA+GMWB contract with high water mark withdrawal benefits
- Characterize the price and worst case policy holder behavior through a Hamilton–Jacobi–Bellman equation (partial differential equation)
- Solve the HJB equation numerically
- Gain insights into ...
  - how the price depends on the risk free rate, volatility, ...
  - the worst case policy holder behavior from the point of view of the issuer
  - how insurance companies should set the fee structure
  - how surrender penalties can be set to disincentivize early surrender
- Derive a hedging strategy

The financial market

Dynamics of an underlying mutual fund

$$dS_t = S_t(\mu dt + \sigma dW_t) = S_t(rdt + \sigma dB_t)$$

The financial market

Dynamics of an underlying mutual fund

$$dS_t = S_t(\mu dt + \sigma dW_t) = S_t(rdt + \sigma dB_t)$$

where

- W is a Brownian motion under the real world probability measure $\mathbb{P}$
- μ is the expected return rate
- σ is the volatility
- B is a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$
- r is a constant risk-free rate

The financial market

Dynamics of an underlying mutual fund

$$dS_t = S_t(\mu dt + \sigma dW_t) = S_t(rdt + \sigma dB_t)$$

where

- W is a Brownian motion under the real world probability measure $\mathbb{P}$
- μ is the expected return rate
- σ is the volatility
- B is a Brownian motion under the risk-neutral probability measure $\mathbb{Q}$
- r is a constant risk-free rate

Market filtration

$$\mathbb{F} = (\mathcal{F}_t) \quad \text{generated by } (S_t)$$

The contract

- Fixed maturity T, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)

The contract

- Fixed maturity T, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
- At $t = 0$, the policy holder pays a premium $c = c_1 + c_2$

The contract

- Fixed maturity T, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
- At $t = 0$, the policy holder pays a premium $c = c_1 + c_2$
- The issuer sets up an account A_t by investing c_1 in the mutual fund, keeps c_2 as a commission and charges fees at rate qA_t

The contract

- Fixed maturity T, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
- At $t = 0$, the policy holder pays a premium $c = c_1 + c_2$
- The issuer sets up an account A_t by investing c_1 in the mutual fund, keeps c_2 as a commission and charges fees at rate qA_t
- The holder chooses a withdrawal rate w_t and a surrender time $\theta \leq T$

The contract

- Fixed maturity T, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
- At $t = 0$, the policy holder pays a premium $c = c_1 + c_2$
- The issuer sets up an account A_t by investing c_1 in the mutual fund, keeps c_2 as a commission and charges fees at rate qA_t
- The holder chooses a withdrawal rate w_t and a surrender time $\theta \leq T$
- If A_t hits zero, the annuity account will be frozen at zero

The contract

- Fixed maturity T, e.g. 10, 15 or 20 years
  (if the policy holder dies, the policy is transferred to a beneficiary)
- At $t = 0$, the policy holder pays a premium $c = c_1 + c_2$
- The issuer sets up an account A_t by investing c_1 in the mutual fund, keeps c_2 as a commission and charges fees at rate qA_t
- The holder chooses a withdrawal rate w_t and a surrender time $\theta \leq T$
- If A_t hits zero, the annuity account will be frozen at zero

Annuity account

$$dA_t = ((r - q)A_t - w_t)dt + \sigma A_t dB_t, \quad A_0 = c_1$$

with absorption at 0

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \leq s \leq t} A_t$

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \leq s \leq t} A_t$
- **Withdrawal rate** w_t must be $\mathbb{H}$-adapted

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \leq s \leq t} A_t$
- **Withdrawal rate** w_t must be $\mathbb{H}$-adapted
- **Constraints and withdrawal fees**
  - When withdrawing below αM_t, fees are charged at rate p_1

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \leq s \leq t} A_t$
- **Withdrawal rate** w_t must be $\mathbb{H}$-adapted
- **Constraints and withdrawal fees**
  - When withdrawing below αM_t, fees are charged at rate p_1
  - For $\alpha M_t < w_t \leq \beta M_t$, fees are charged at a higher rate p_2

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \leq s \leq t} A_t$
- **Withdrawal rate** w_t must be $\mathbb{H}$-adapted
- **Constraints and withdrawal fees**
    - When withdrawing below αM_t, fees are charged at rate p_1
    - For $\alpha M_t < w_t \leq \beta M_t$, fees are charged at a higher rate p_2
    - Constraint on total withdrawal: $\int_0^t w_s ds \leq \gamma M_t$

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \le s \le t} A_t$
- **Withdrawal rate** w_t must be $\mathbb{H}$-adapted
- **Constraints and withdrawal fees**
  - When withdrawing below αM_t, fees are charged at rate p_1
  - For $\alpha M_t < w_t \le \beta M_t$, fees are charged at a higher rate p_2
  - Constraint on total withdrawal: $\int_0^t w_s ds \le \gamma M_t$
- **Guaranteed withdrawal** If the account hits 0, withdrawal is still allowed.

Withdrawal

Privat information

Privat events such as changes in employment status, health or family situation is modeled with a filtration $\mathbb{G} = (\mathcal{G}_t)$

Assume: $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$

Information of the policy holder: $\mathbb{H} = \mathbb{F} \vee \mathbb{G}$

- **High water mark** $M_t = \sup_{0 \leq s \leq t} A_t$
- **Withdrawal rate** w_t must be $\mathbb{H}$-adapted
- **Constraints and withdrawal fees**
  - When withdrawing below αM_t, fees are charged at rate p_1
  - For $\alpha M_t < w_t \leq \beta M_t$, fees are charged at a higher rate p_2
  - Constraint on total withdrawal: $\int_0^t w_s ds \leq \gamma M_t$
- **Guaranteed withdrawal** If the account hits 0, withdrawal is still allowed.

Income rate

$$f(w_t, M_t) = (1 - p_1)\min(w_t, \alpha M_t) + (1 - p_2)\max(w_t - \alpha M_t, 0), \quad w_t \leq \beta M_t$$

Surrender

- The holder can surrender the policy at an $\mathbb{H}$-stopping time $\theta \leq T$

Surrender

- The holder can surrender the policy at an $\mathbb{H}$-stopping time $\theta \leq T$
- At time θ the issuer charges a penalty of $k(\theta)A_\theta$, returns $(1 - k(\theta))A_\theta$ to the holder and terminates the contract,

  where $k : [0, T] \to [0, 1]$ is a surrender penalty function

Surrender

- The holder can surrender the policy at an $\mathbb{H}$-stopping time $\theta \leq T$
- At time θ the issuer charges a penalty of $k(\theta)A_\theta$, returns $(1 - k(\theta))A_\theta$ to the holder and terminates the contract,

  where $k : [0, T] \to [0, 1]$ is a surrender penalty function
- Typically, $k(T) = 0$

Policy holder behavior

- The policy holder chooses the withdrawal strategy w and surrender time θ

Policy holder behavior

- The policy holder chooses the withdrawal strategy w and surrender time θ
- Worst expected cost of the payments faced by the issuer

$$
\begin{aligned}
D(A_0) &= \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^{\mathbb{Q}} e^{-r\theta} k(\theta) A_{\theta} \\
&\quad - \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t>0\}} e^{-rs} \left[qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0) \right] ds
\end{aligned}
$$

Policy holder behavior

- The policy holder chooses the withdrawal strategy w and surrender time θ
- Worst expected cost of the payments faced by the issuer

$$
\begin{aligned}
D(A_0) \quad = \quad & \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^{\mathbb{Q}} e^{-r\theta} k(\theta) A_\theta \\
& - \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t>0\}} e^{-rs} \left[qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0) \right] ds
\end{aligned}
$$

- One has $A_0 + D(A_0) = E(A_0)$ for

$$
E(A_0) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\theta} e^{-rs} f(w_s, M_s) ds + e^{-r\theta} (1 - k(\theta)) A_\theta \right]
$$

worst case policy holder behavior ... not actual policy holder behavior!

Policy holder behavior

- The policy holder chooses the withdrawal strategy w and surrender time θ
- Worst expected cost of the payments faced by the issuer

$$
\begin{aligned}
D(A_0) \quad = \quad & \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^{\mathbb{Q}} e^{-r\theta} k(\theta) A_\theta \\
& - \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t>0\}} e^{-rs} \left[q A_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0) \right] ds
\end{aligned}
$$

- One has $A_0 + D(A_0) = E(A_0)$ for

$$
E(A_0) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\theta} e^{-rs} f(w_s, M_s) ds + e^{-r\theta} (1 - k(\theta)) A_\theta \right]
$$

worst case policy holder behavior ... not actual policy holder behavior!

- The policy is correctly priced (from the issuer's perspective) if

$$
D(c_1) = c_2 \quad \Leftrightarrow \quad E(c_1) = c_1 + c_2
$$

Policy holder behavior

- The policy holder chooses the withdrawal strategy w and surrender time θ
- Worst expected cost of the payments faced by the issuer

$$\begin{aligned} D(A_0) \quad = \quad & \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^{\mathbb{Q}} e^{-r\theta} k(\theta) A_\theta \\ & - \mathbb{E}^{\mathbb{Q}} \int_0^{\theta} 1_{\{A_t>0\}} e^{-rs} \left[qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0) \right] ds \end{aligned}$$

- One has $A_0 + D(A_0) = E(A_0)$ for

$$E(A_0) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_0^{\theta} e^{-rs} f(w_s, M_s) ds + e^{-r\theta} (1 - k(\theta)) A_\theta \right]$$

  worst case policy holder behavior ... not actual policy holder behavior!

- The policy is correctly priced (from the issuer's perspective) if

$$D(c_1) = c_2 \quad \Leftrightarrow \quad E(c_1) = c_1 + c_2$$

- **Good news:** $E(A_0)$ is attained for w, θ only depending on market information $\mathbb{F}$

Policy holder behavior

- The policy holder chooses the withdrawal strategy w and surrender time θ
- Worst expected cost of the payments faced by the issuer

$$D(A_0) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \int_0^\theta 1_{\{A_t=0\}} e^{-rs} f(w_s, M_s) ds - \mathbb{E}^{\mathbb{Q}} e^{-r\theta} k(\theta) A_\theta$$
$$-\mathbb{E}^{\mathbb{Q}} \int_0^\theta 1_{\{A_t>0\}} e^{-rs} \left[qA_t + p_1 \min(w_t, \alpha M_t) + p_2 \max(w_t - \alpha M_t, 0) \right] ds$$

- One has $A_0 + D(A_0) = E(A_0)$ for

$$E(A_0) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_0^\theta e^{-rs} f(w_s, M_s) ds + e^{-r\theta}(1 - k(\theta)) A_\theta \right]$$

worst case policy holder behavior ... not actual policy holder behavior!

- The policy is correctly priced (from the issuer's perspective) if

$$D(c_1) = c_2 \quad \Leftrightarrow \quad E(c_1) = c_1 + c_2$$

- **Good news:** $E(A_0)$ is attained for w, θ only depending on market information $\mathbb{F}$
- **Bad news:** The optimization problem $E(A_0)$ is not Markovian

Adding more state variables

For given $0 \leq t \leq T$, $0 \leq a \leq m$, $0 \leq z \leq \gamma m$, w, define

$$
\begin{aligned}
dA_s^{t,a,w} &= ((r-m)A_s^{t,a,w} - w_s)ds + \sigma A_s^{t,a,w} dB_s, \quad A_t^{t,a,w} = a \\
M_s^{t,a,g,w} &= m \vee \sup_{t \leq u \leq s} A_u^{t,a,m,w} \\
dZ_s^{t,z,w} &= w_s ds, \quad Z_t^{t,z,w} = z
\end{aligned}
$$

Adding more state variables

For given $0 \leq t \leq T$, $0 \leq a \leq m$, $0 \leq z \leq \gamma m$, w, define

$$
\begin{aligned}
dA_s^{t,a,w} &= ((r-m)A_s^{t,a,w} - w_s)ds + \sigma A_s^{t,a,w} dB_s, \quad A_t^{t,a,w} = a \\
M_s^{t,a,g,w} &= m \vee \sup_{t \leq u \leq s} A_u^{t,a,m,w} \\
dZ_s^{t,z,w} &= w_s ds, \quad Z_t^{t,z,w} = z
\end{aligned}
$$

A "non-standard" standard stochastic control problem

$$
V(t,a,m,z) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T e^{-r(s-t)} f(w_s, M_s^{t,a,m,w}) ds + e^{-r(\theta-t)}(1-k(\theta)) A_\theta^{t,a,w} \right]
$$

where w and θ are adapted to $\mathcal{F}_s^t = \sigma(B_s - B_t)$, $s \in [t, T]$

Adding more state variables

For given $0 \le t \le T$, $0 \le a \le m$, $0 \le z \le \gamma m$, w, define

$$\begin{aligned} dA_s^{t,a,w} &= ((r-m)A_s^{t,a,w} - w_s)ds + \sigma A_s^{t,a,w} dB_s, \quad A_t^{t,a,w} = a \\ M_s^{t,a,g,w} &= m \vee \sup_{t \le u \le s} A_u^{t,a,m,w} \\ dZ_s^{t,z,w} &= w_s ds, \quad Z_t^{t,z,w} = z \end{aligned}$$

A "non-standard" standard stochastic control problem

$$V(t,a,m,z) = \sup_{w,\theta} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T e^{-r(s-t)} f(w_s, M_s^{t,a,m,w}) ds + e^{-r(\theta-t)}(1-k(\theta)) A_\theta^{t,a,w} \right]$$

where w and θ are adapted to $\mathcal{F}_s^t = \sigma(B_s - B_t)$, $s \in [t,T]$

One has $V(0,a,a,0) = E(a)$

The HJB equation

Theorem

$V(t, a, m, z)$ is a viscosity solution of

$$
\begin{aligned}
\min(-v_t - H(a, m, v, v_a, v_z, v_{aa}), v - (1-k)a) &= 0 \text{ for } 0 < a < m,\ 0 \le z < \gamma m \\
\min(-v_t - H_0(a, v, v_a, v_{aa}), v - (1-k)a) &= 0 \text{ for } 0 < a < m,\ z = \gamma m \\
v(t, 0, m, z) &= \psi(t, m, z) \\
v_m(t, m, m, z) &= 0 \\
v(T, a, m, z) &= a,
\end{aligned}
$$

where

$$
H(a, m, z, v, v_a, v_z, v_{aa}) =
$$

$$
\sup_{0 \le w \le \beta m} \{f(w, m) + w(v_z - v_a)\} - rv + (r-q)av_a + \frac{1}{2}\sigma^2 a^2 v_{aa}
$$

$$
H_0(a, v, v_a, v_{aa}) = -rv + (r-q)av_a + \frac{1}{2}\sigma^2 a^2 v_{aa}
$$

$$
\psi(t, m, z) = \sup_{0 \le w \le \beta m} \int_t^T e^{-r(s-t)} f(w_s, m)\,ds \quad \text{such that} \int_t^T w_s\,ds \le \gamma m - z.
$$

Non-linear parabolic PDE on $[0, T] \times \mathbb{R}^3$ with a free boundary and unusual boundary conditions

Reducing the dimension

Theorem

$V(t, a, m, z) = mW(t, a/m, z/m) = mW(t, x, y)$, where W is a viscosity solution of

$$\begin{aligned}
\min(-v_t - G(x, y, v, v_x, v_y, v_{xx}), v - (1-k)x) &= 0 \quad \text{for } (x, y) \in (0, 1) \times [0, \gamma) \\
\min(-v_t - G_0(x, v, v_x, v_{xx}), v - (1-k)x) &= 0 \quad \text{for } (x, y) \in (0, 1) \times \{\gamma\} \\
v(t, 0, y) &= \zeta(t, y) \\
v_x(t, 1, y) + y v_y(t, 1, y) &= v(t, 1, y) \\
v(T, x, y) &= x,
\end{aligned}$$

where

$$G(x, y, v, v_x, v_y, v_{xx}) =$$

$$\sup_{0 \le u \le \beta} \{u(v_y - v_x) + f(u, 1)\} - rv + (r-q)xv_x + \frac{1}{2}\sigma^2 x^2 v_{xx}$$

$$G_0(x, v, v_x, v_{xx}) = -rv + (r-q)xv_x + \frac{1}{2}\sigma^2 x^2 v_{xx}$$

$$\zeta(t, y) = \sup_{0 \le u \le \beta} \int_t^T e^{-r(s-t)} f(u_s, 1) ds \quad \text{such that} \int_t^T u_s ds \le \gamma - y.$$

Non-linear parabolic PDE on $[0, T] \times [0, 1] \times [0, \gamma]$ with a free boundary and (even more) unusual boundary conditions

Worst case strategy

- The worst case withdrawal strategy $\hat{w}(t, A_t, M_t, Z_t)$ is given by the maximizer of

$$w \mapsto f(w, M_t) + w \left[V_z(t, A_t, M_t, Z_t) - V_a(t, A_t, M_t, Z_t)\right]$$

Worst case strategy

- The worst case withdrawal strategy $\hat{w}(t, A_t, M_t, Z_t)$ is given by the maximizer of

$$w \mapsto f(w, M_t) + w\,[V_z(t, A_t, M_t, Z_t) - V_a(t, A_t, M_t, Z_t)]$$

- The worst case surrender time is

$$\theta = \inf\{t \geq 0 : V(t, A_t, M_t, Z_t) = (1 - k(t))A_t\}$$

Worst case strategy

- The worst case withdrawal strategy $\hat{w}(t, A_t, M_t, Z_t)$ is given by the maximizer of

$$w \mapsto f(w, M_t) + w\left[V_z(t, A_t, M_t, Z_t) - V_a(t, A_t, M_t, Z_t)\right]$$

- The worst case surrender time is

$$\theta = \inf\{t \geq 0 : V(t, A_t, M_t, Z_t) = (1 - k(t))A_t\}$$

- The issuer can set $k(t)$ so that the worst case policy holder never surrenders early

Numerical scheme

- Semi-Lagrangian scheme with an obstacle
- Backwards in time
- Solves an optimization problem in every time-step
- Converges to the true solution if the mesh size of the discretization goes to zero
- Gives approximations to $V(t, a, m, z)$ and the worst case behavior $\hat{w}$ and $\hat{\theta}$

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t > 0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t>0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
- Make payments at rate $f(w_t, M_t)$

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t>0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
- Make payments at rate $f(w_t, M_t)$
- Keep the rest of the portfolio value in the money market account

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t>0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
- Make payments at rate $f(w_t, M_t)$
- Keep the rest of the portfolio value in the money market account
- Pay out $(1 - k(\theta))A_\theta$ at the surrender time θ

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t>0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
- Make payments at rate $f(w_t, M_t)$
- Keep the rest of the portfolio value in the money market account
- Pay out $(1 - k(\theta))A_\theta$ at the surrender time θ
- This will super-hedge the contract

Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t > 0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
- Make payments at rate $f(w_t, M_t)$
- Keep the rest of the portfolio value in the money market account
- Pay out $(1 - k(\theta))A_\theta$ at the surrender time θ
- This will super-hedge the contract
- It will exactly hedge the contract if (w, θ) equals the worst case strategy $(\hat{w}, \hat{\theta})$

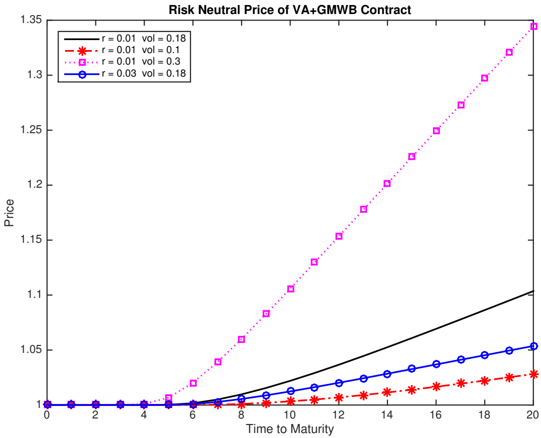
Hedging

The issuer can super-hedge the contract by trading in S_t and the money market account

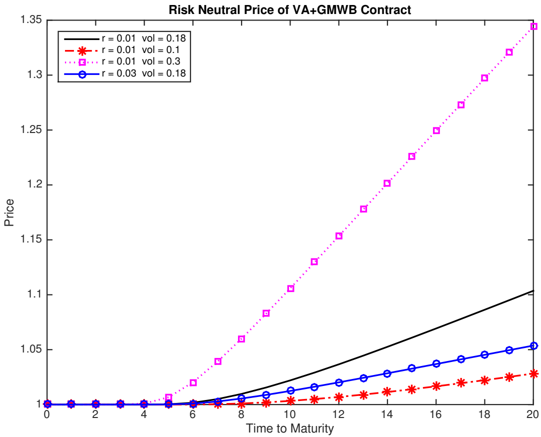
- Assume $V(0, c_1, c_1, 0) = E(c_1) = c_1 + c_2$
- Start a hedging portfolio with initial capital $c_1 + c_2$
- Observe S_t together with the actual withdrawal strategy w_t and surrender time θ
- Hold the amount $1_{\{A_t>0\}} A_t V_a(t, A_t, M_t, Z_t)$ in the mutual fund
- Make payments at rate $f(w_t, M_t)$
- Keep the rest of the portfolio value in the money market account
- Pay out $(1 - k(\theta))A_\theta$ at the surrender time θ
- This will super-hedge the contract
- It will exactly hedge the contract if (w, θ) equals the worst case strategy $(\hat{w}, \hat{\theta})$

Proof: Itô's formula

Numerical results

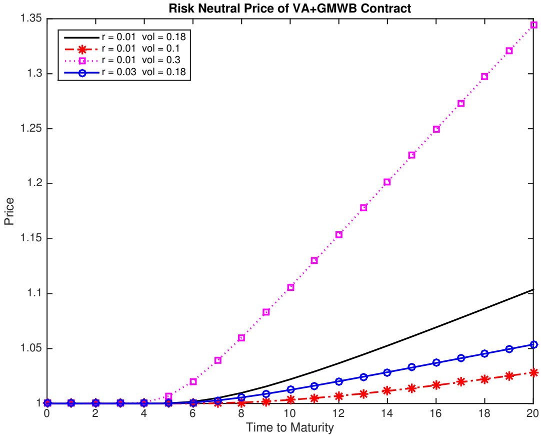


Numerical results



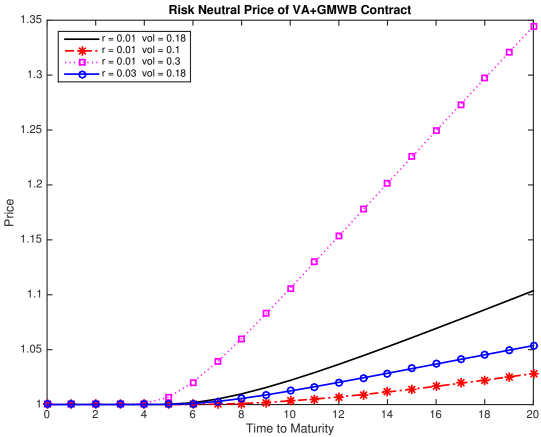
- Price is increasing in σ

Numerical results



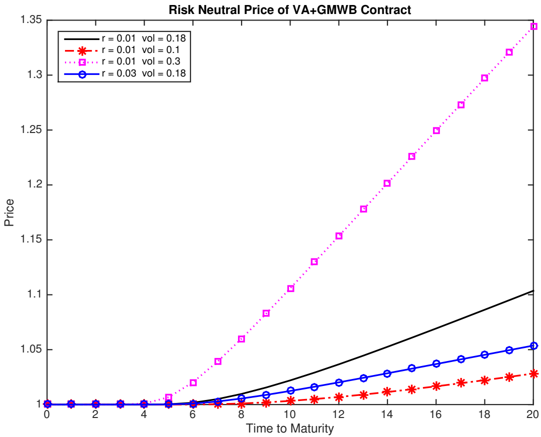
- Price is increasing in σ ... not taken into account by insurance companies

Numerical results



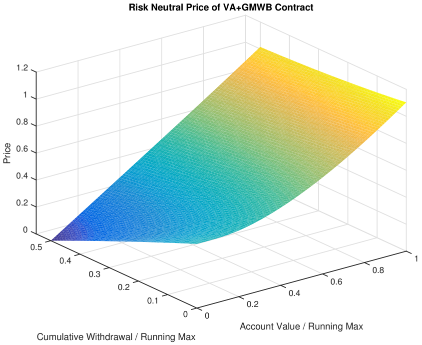
- Price is increasing in σ ... not taken into account by insurance companies
- Price is decreasing in r

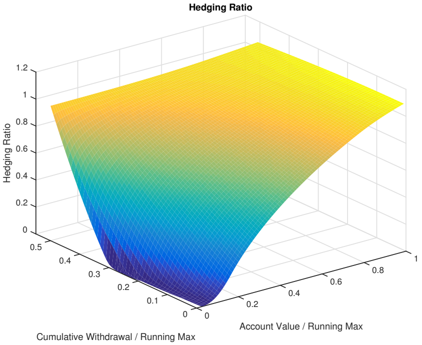
Numerical results



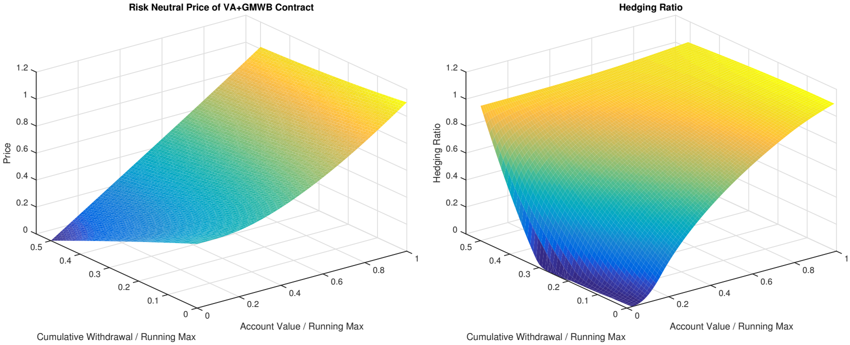
- Price is increasing in σ ... not taken into account by insurance companies
- Price is decreasing in r ... makes these products difficult to sell in a low interest rate environment

Price and hedging ratio as functions of $x = a/m$ and $y = z/m$



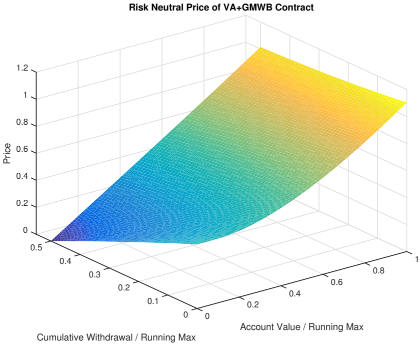


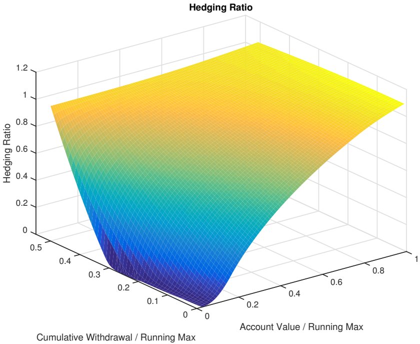
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$



- Price is increasing in $x = a/m$ and decreasing in $y = z/m$

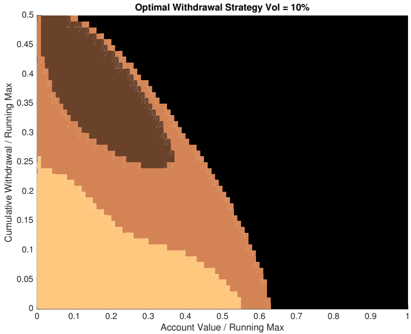
Price and hedging ratio as functions of $x = a/m$ and $y = z/m$





- Price is increasing in $x = a/m$ and decreasing in $y = z/m$
- The hedge is always long in S_t in contrast to the hedge of a put option)

Worst case withdrawal and surrender



- $T - t = 5$ years, $r = 1\%$, $\sigma = 18\%$, $q = 0.8\%$, $p_1 = 0$, $p_2 = 5\%$, $k \equiv 0$
- yellow = full withdrawal (at rate βM_t)
- light brown = intermediate withdrawal (at rate αM_t)
- brown = no withdrawal
- black = surrender

Discouraging early surrender

Set the surrender penalty function such that

$$k(t) \geq (T - t)q$$

Merci!